

# SHM

## Simple Harmonic Motion

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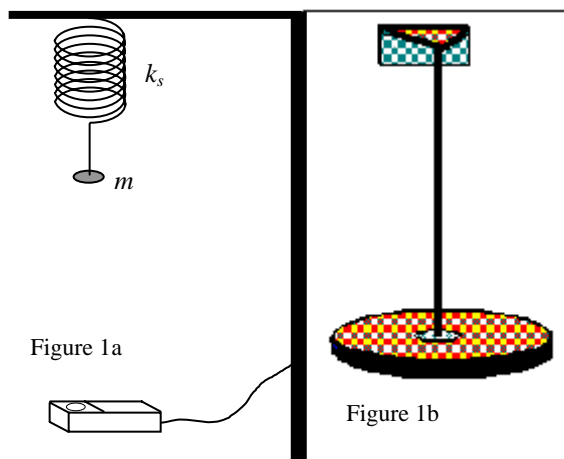
### Learning Objectives:

During this lab, you will

1. communicate scientific results in writing.
2. estimate the uncertainty in a quantity that is calculated from quantities that are uncertain.
3. learn how to use a histogram to analyze a data set.
4. fit a function more complicated than a straight line to a data set.
5. test a physical law experimentally.

### A. Introduction

A body will oscillate in linear simple harmonic motion if it is acted upon by a restoring force whose magnitude is proportional to the linear displacement of the body from its equilibrium position ( $F = -kx$ ). Similarly, a body will oscillate in rotational simple harmonic motion if there is a restoring torque that is proportional to the angular displacement of the body from its equilibrium position ( $\tau = -k\theta$ ). For this experiment, you



**Figure 1:** Schematics for experimental setups. a) Spring-mass (linear) oscillator. b) Torsion pendulum (rotational) oscillator.

will explore both kinds of harmonic motion.

You will use the most common example of *linear* simple harmonic motion, an object attached to a spring (see Fig. 1a). The most common example of *rotational* simple harmonic motion is the simple pendulum where, for small angular displacements of the pendulum from the equilibrium (*vertical*) position, the restoring torque is approximately proportional to the angle of displacement.

We have chosen another example for this experiment, the torsion pendulum. A long cylindrical metal rod is held fixed in a support bracket with a heavy cylindrical plate attached to its lower end (Figure 1b). If you rotate the plate through an angle  $\theta$ , the rod will exert on the plate a restoring torque that is proportional to  $\theta$ , as long as  $\theta$  does not exceed the elastic limit of the rod. If you rotate the plate and release it, the plate will oscillate in simple harmonic motion.

From measurements of the period of the oscillation and the geometry of the rod and plate, you can calculate the *torsion modulus*, a quantity that characterizes the resistance of a material to twisting.

You must write a report for this lab.

### B. Apparatus

The two setups are a spring and a torsion pendulum. The spring apparatus includes hanging masses suspended from a spring attached to a ring stand. A sonic motion detector interfaced to a computer running *LoggerPro* determines the position of the hanging mass.

The torsion pendulum is a metal rod, mounted in a wall bracket, with a metal plate and cylinder. You will also use a meter stick, calipers, computer and *Logger Pro*. Most of the pendula are constructed with aluminum plates, but a few use either grey- or blue-painted iron. The masses of each type of plate are in Table 1.

**Table 1. Plate masses**

Aluminum	4.7±0.1 kg
Grey Iron	4.6±0.1 kg
Blue Iron	4.8±0.1 kg

Neighboring groups will share each apparatus for this experiment. Each group must acquire and analyze their own data (but partners within a group may share data). Data acquisition is very quick and easy, so sharing a setup should not be a problem.

## C. Theory

### C.1. Translational Simple Harmonic Motion

If a spring is extended (or compressed) to a position  $x$  different from its equilibrium position  $x_0$ , it exerts a restoring force

$$F = -k(x-x_0), \quad (1)$$

where  $k$  is the *spring constant*. An object of mass  $m$  attached to the spring therefore undergoes an acceleration

$$a = \frac{F}{m} = \frac{-k_s(x-x_0)}{m} \quad (2)$$

The object thus obeys the equation of motion

$$-k_s(x-x_0) = m \frac{d^2x}{dt^2}, \quad (3)$$

which has the general solution

$$x = x_m \sin(\omega t + \phi) + x_0, \quad (4)$$

with

$$\omega = \sqrt{k_s/m}. \quad (5)$$

Equation 4 describes an oscillation where  $x_m$ , the amplitude of the oscillation, is half the peak-to-peak variation in the position. The offset  $x_0$  is the relaxed position of the spring.  $\omega$  (*lower-case omega*) is the angular frequency which is related to the frequency  $f$  and period  $T$  of the oscillation by  $\omega = 2\pi f = 2\pi/T$ . The angular frequency  $\omega$  can therefore be determined from a measurement of the average period of many oscillations.  $\phi$  (*lower-case phi*) is the phase angle of the

oscillation. The phase angle  $\phi$  can be determined from where the oscillation is at  $t=0$ . For example, the phase  $\phi = 0$  if at  $t=0$  the oscillation is at its equilibrium position  $x_0$  and the position  $x$  initially increases with time;  $\phi = \pi$  if at  $t=0$  the oscillation is at its equilibrium position  $x_0$  and the position  $x$  initially decreases with time.

### C.2. Rotational Simple Harmonic Motion

Similarly, if an elastic rod is twisted through an angle  $\theta$ , the restoring torque  $\tau$  is given by the expression

$$\tau = -k_t \theta \quad (6)$$

where  $k$  is the *torsion constant* of the rod. Combining this expression with Newton's second law in rotational motion form,

$$\tau = I\alpha, \quad (7)$$

gives

$$-k_t \theta = I\alpha \quad (8)$$

for the equation of motion of the rod and plate. In this expression  $I$  is the moment of inertia of the rotating object and  $\alpha = d^2 \theta / dt^2$  is the angular acceleration of the system. The equation of motion is thus

$$-k_t \theta = I \frac{d^2 \theta}{dt^2} \quad (9)$$

The general solution to this equation is

$$\theta = \theta_m \sin(\omega t + \phi) \quad (10)$$

where  $\theta_m$  is the amplitude of the oscillation,  $\omega$  is the angular frequency ( $\omega = 2\pi f = 2\pi/T$  where  $f$  and  $T$  are the frequency and period of the oscillation), and  $\phi$  is the phase angle. Note that neither  $\phi$  nor  $\omega$  refer to physical angles. Only  $\theta(t)$ , the angle of rotation, and its amplitude  $\theta_m$  can be measured with a protractor. The phase angle  $\phi$  and the angular frequency  $\omega$  describe how the oscillation varies with time. In particular, the symbol  $\omega$  should not be confused with the rate of change of the displacement  $\Delta\theta/\Delta t$ , the angular velocity, which normally is given the same symbol,  $\omega$ .

In order that Equation 10 be a solution of Equation 9, the following relation must be obeyed:

$$\omega = \sqrt{k_t/I}. \quad (11)$$

Thus, from measurements of the period  $T$  of oscillation and the moment of inertia  $I$  of the plate, you can determine the torsion constant  $k_t$  of the rod. If your circular plate is a solid cylinder,

$$I = \frac{1}{2}MR^2 \quad (12)$$

where  $M$  and  $R$  are the mass and radius of the plate. You will neglect the moment of inertia of the rod because it is negligible compared to the plate's (*the rod has a much smaller mass and radius*).

Note that the formulae for rotational harmonic motion are exactly analogous to the formulae for *translational* harmonic motion of a linear spring given above.

### C.3. Torsion Modulus

The torsion constant  $k_t$  depends upon the shape of the rod as well as on the material of which it is made. The relation is

$$k_t = \frac{nA^2}{2\pi L} \quad (13)$$

where  $n$  is the torsion modulus,  $A$  is the cross-sectional area of the rod and  $L$  is the length of the rod. This equation predicts, as you might expect, that a long, skinny rod would be much easier to rotate than a short, fat one.

## D. Procedure

### D.1 Translational Simple Harmonic Motion

Record the mass of your hanger. Place the sonic motion detector beneath your spring. Start *LoggerPro* and open the file *P:LoggerPro 3\\_Mech Labs\SHM*. Move some flat object up and down above the detector in order to determine over what range the detector reads positions.

First take data for a small-amplitude oscillation. Attach the hanger and pull it a few cm from its equilibrium position, being

careful to prevent the hanger from oscillating side to side. Watch the sine-wave oscillation that results. Try to estimate each of the parameters of the oscillation described by Eq. 4. Test the accuracy of your estimates by using *LoggerPro* to plot the oscillation corresponding to your parameters. Select DATA / NEW CALCULATED COLUMN and type in the equation

$$x_m * \sin(\omega * \text{“Time”} + \phi) + x_0,$$

where you replace the parameters  $x_m$ ,  $\omega$ ,  $\phi$ , and  $x_0$  with your estimated values (*make sure to type all leading zeros; LoggerPro may not understand that “.45” means “0.45”*). Press OK. To view the data and your manual fit at once, select OPTIONS / GRAPH OPTIONS / Axes Options and choose both columns under Y axis. Select DONE to see the resulting oscillation. If the resulting curve looks different enough from your data that you must have made an error, select DATA / COLUMN OPTIONS / Calculated Column to correct your parameters appropriately.

The fitting routine in *LoggerPro* makes it easy to estimate the parameters accurately, so use it to check your fit, and use the routine for all further measurements without first making a manual fit. Select ANALYZE / CURVE FIT, choose to fit a Sine Wave (at the bottom of the scroll bar) to your “Latest: Position” data, and click the TRY FIT button. You should see a well-fit sine wave. If the sine wave does not fit your data, you may need to adjust your Region of Interest before performing the fit. Print the plot and include it in your report.

Although *LoggerPro* returns a quantity describing the goodness of the fit, it does not return the uncertainty on the parameter values. To estimate the uncertainty on the angular frequency  $\omega$ , repeat data-taking and the *LoggerPro* fit for 2-4 more trials at small amplitude and record the results of each fit. You should first delete your manual fit by

selecting DATA/Delete Column/Calculated Column.

Next, take data to test the dependence of  $\omega$  on the amplitude of oscillations and on the object's mass. First take data with a large-amplitude oscillation (about 10 cm) using the same mass, and record the best-fit angular frequency  $\omega_2$ . Then add 30-50 g to the hanger, adjusting the height of the spring if necessary so that the motion will still be in range of the detector. Record the motion of oscillations and determine the best-fit angular frequency  $\omega_3$ .

### D.2. Torsion Pendulum (Rotational SHM)

Measure the diameter of the plate and the length of the rod. (*You should measure the rod's "free length;" you don't have to worry about parts of the rod inside thicker objects at either end of its length.*) Use the Vernier calipers to make an accurate measurement of the diameter of the rod. Seek help in using the Vernier scale, if necessary. The uncertainty in the diameter should be no more than 0.2 mm; it's important to minimize this error because it enters as the 4<sup>th</sup> power in your calculation of the torsion modulus.

Start *LoggerPro* and open the file *P:Logger Pro 3\Mech Labs\T\_PEND*. Adjust the position of the photogate so that the computer accurately records each period of the oscillation.

Give the plate a twist to set it into oscillation. Don't twist too far and risk damaging the rod. An arrow is drawn on the plate at a 45° angle; don't twist past this mark. Place your fingers loosely around the rod close to the plate to guide the rod for a few oscillations so that it doesn't also oscillate with a pendulum motion. Use *Logger Pro* to time at least 25 periods. Translational oscillations will make the measured period oscillate around a fixed value. Retake your data if oscillations in the period appear large.

## E. Analysis

### E.1. Spring

Using the best-fit angular frequencies for your first trials with small-amplitude oscillations and just the hanger, determine your best measurement of the angular frequency  $\omega$ , the estimated uncertainty on a single measurement  $\sigma$ , and the error in the mean  $\delta$ . Use the formulae in Appendix V.B.1 or type your results into *Origin* and use select ANALYSIS/STATISTICS ON COLUMNS to calculate the results. Does the best-fit angular frequency for large-amplitude oscillations agree with your best measurement of  $\omega$ ?

Use Eq. 5 with your measured values of the hanging mass and the angular frequency to determine the spring constant  $k_s$  and its uncertainty. Use this value to predict the angular frequency of oscillations  $\omega_p$  (and its uncertainty) with the increased mass. Does  $\omega_p$  agree with your measured value  $\omega_3$ ?

### E.2. Torsion Pendulum

Make a histogram of the measured periods of oscillation of the plate. To do this, import the data into ORIGIN, make sure neither column is highlighted and click PLOT / STATISTICAL GRAPHS - HISTOGRAM. You can change the ranges and intervals by clicking on the plot legend to call up the PLOT DETAILS window and its DATA tab. See Appendix VII for a discussion of histograms.

Obtain the mean period ( $T$ ), standard deviation ( $\sigma$ ), and the standard error ( $\delta$ ) of the data. Return to the data window and highlight the y column. Click on STATISTICS / DESCRIPTIVE STATISTICS / STATISTICS ON COLUMNS. The values of the mean period, standard deviation, and standard error on the mean will pop up in a window. (*See Appendix V.B.1 if you are uncertain what*

these quantities mean.) Copy them and paste them into your original histogram.

To visually check that these results make sense, add a Gaussian curve, with the correct mean and standard deviation, to the histogram. First, be certain that your histogram window is highlighted. Select GRAPH / ADD FUNCTION TO GRAPH. In the box that pops up, type in a Gaussian function that describes your data as

$$Max \cdot \exp(-\frac{(x - T)^2}{2 \cdot \sigma^2})$$

where  $Max$  is the maximum number of counts in a bin and  $T$  and  $\sigma$  are the mean and standard deviation, respectively.

Edit your histogram for proper viewing (add a title, edit the axes, etc.), save your project and print out copies to include in your reports. If the Gaussian curve doesn't fit your data well, try to explain why.

Assume that the plate is a solid cylinder and calculate its moment of inertia  $I_P$ . Calculate the torsion constant  $k_t$  from  $I_P$  and  $T$ . Calculate the torsion modulus  $n$  from

Equation 13. Show that a 1% error in the diameter will give a 4% error in the torsion modulus calculated from Equation 13.

Table 2 includes accepted values of the torsion moduli for a variety of materials (*from the CRC Handbook of Chemistry and Physics.*) Based on your calculated torsion modulus and its uncertainty, what material(s) could the rod be? If you doubt your results, explain why.

**Table 2. Torsion moduli**

Lead	$0.54 \times 10^{10} \text{ N}\cdot\text{m}^{-2}$
Magnesium	$1.67 \times 10^{10} \text{ N}\cdot\text{m}^{-2}$
Aluminum	$2.37 \times 10^{10} \text{ N}\cdot\text{m}^{-2}$
Brass	$3.53 \times 10^{10} \text{ N}\cdot\text{m}^{-2}$
Copper	$4.24 \times 10^{10} \text{ N}\cdot\text{m}^{-2}$
Iron	$7.0 \times 10^{10} \text{ N}\cdot\text{m}^{-2}$
Nickel	$7.0-7.6 \times 10^{10} \text{ N}\cdot\text{m}^{-2}$
Steel	$7.8 \times 10^{10} \text{ N}\cdot\text{m}^{-2}$
Molybdenum	$14.7 \times 10^{10} \text{ N}\cdot\text{m}^{-2}$
Tungsten	$14.8 \times 10^{10} \text{ N}\cdot\text{m}^{-2}$

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